Approximation of the random inertial manifold of singularly perturbed stochastic wave equations

Yan Lv* Wei Wang† A. J. Roberts‡

September 4, 2012

Abstract

By applying Rohlin's result on the classification of homomorphisms of Lebesgue space, the random inertial manifold of a stochastic damped nonlinear wave equations with singular perturbation is proved to be approximated almost surely by that of a stochastic nonlinear heat equation which is driven by a new Wiener process depending on the singular perturbation parameter. This approximation can be seen as the Smolukowski–Kramers approximation as time goes to infinity. However, as time goes infinity, the approximation changes with the small parameter, which is different from the approximation on a finite interval.

Keywords Random inertial manifold, singularly perturbed stochastic wave equation, Lebesgue space, homomorphism,

Mathematics Subject Classifications (2000) 60F10, 60H15, 35Q55.

1 Introduction

The motion of particles in a continuum with a stochastic force $\dot{W}$, by Newton's law, is assumed to be described by the following stochastically forced damped wave equation (SWE) [7]

$$\nu u^\nu_{tt}(t,x) + u^\nu_t(t,x) = \Delta u^\nu + f(u^\nu(t,x)) + \sigma W_t(t,x), \quad t>0\,,\ x\in D\,, \tag{1}$$

$$u^\nu(0,x)=u_0\,,\quad u^\nu_t(0,x)=u_1\,,\quad u(t,x)=0\,,\quad x\in\partial D\,. \tag{2}$$

*School of Science, Nanjing University of Science & Technology, Nanjing, 210094, China mailto:lvyan1998@yahoo.com.cn

†Department of Mathematics, Nanjing University, Nanjing, 210093, China mailto:wangweinju@yahoo.com.cn; and School of Mathematical Sciences, University of Adelaide, South Australia, Australia

‡School of Mathematical Sciences, University of Adelaide, South Australia, Australia mailto:anthony.roberts@adelaide.edu.au

Here we consider the problem on a one dimensional bounded spatial domain and for simplicity we assume the domain $D = (0, \pi)$, W is a Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ which is determined later. The small parameter $\nu > 0$ characterises the density of particles.

The Smolukowski–Kramers approximation of infinite dimension [7] states that on any finite time interval $[0, T]$, for $0 < \nu \ll 1$, the solution u^ν to the SWE (1)–(2) is approximated by the solution of the following stochastic nonlinear heat equation (SHE)

$$u_t(t, x) = \Delta u(t, x) + f(u(t, x)) + \sigma W_t(t, x)\,, \quad x \in D\,, \tag{3}$$

$$u(0, x) = u_0\,, \quad u(t, x) = 0\,, \quad x \in \partial D\,, \tag{4}$$

in the sense that

$$\lim_{\nu\to 0} \mathbb{P}\left\{\sup_{0\le t\le T} \|u^\nu(t) - u(t)\|_{L^2(D)} \ge \delta\right\} = 0 \tag{5}$$

for any $\delta > 0$. Here we give an almost sure approximation for the random dynamics of the SWE (1)–(2); that is, we consider the approximation of the long time behaviour of u^ν for small ν. We call this the Smolukowski–Kramers approximation for the SWE (1)–(2) as $t \to \infty$. For this we consider the approximation of random inertial manifold to SWE (1) for small $\nu > 0$.

Random invariant manifolds are very important in modelling random dynamics of a stochastic system [22, e.g.], especially infinite dimensional systems [2, 12, 13, 17, 25, 21, e.g.]. For example, Wang and Roberts [26] showed one way to view spatial discretisations of SPDEs as a stochastic slow invariant manifold. Duan et al. [12, 13] generalized deterministic methods to construct a random invariant manifold for stochastic partial differential equations with multiplicative noise. Roberts [21] established approximations to stochastic slow invariant manifold models of nonlinear reaction-diffusion SPDEs. Then some subsequent work constructed random invariant manifolds for a stochastic wave equation [16, 17, e.g.]. We apply the Lyapunov–Perron method for stochastic partial differential equations [13] to construct a random invariant manifold for the SWE (1)–(2) for any fixed $\nu > 0$ and a random invariant manifold for the SHE (8)–(4). Notice that the noises in systems (1)–(2) and (8)–(4) are additive: to apply the Lyapunov–Perron method to SPDEs with additive noise, we need a stationary solution to transform the SPDEs to a random differential system (14) [13]. Then a random invariant manifold to this stationary solution can be constructed [13]. However, for a nonlinear stochastic system, more detailed estimates on solutions is required to ensure the existence of a stationary solution [10, 11] and such a stationary solution is difficult to be written out explicitly; we do this transform by introducing stationary solutions of some linear systems, which are

written out explicitly, see section 3. For the SWE (1)–(2) we introduce the stationary solution $z^{*\nu}$ solving the linear system

$$\nu z^{\nu}_{tt} + z^{\nu}_{t} = \Delta z^{\nu} + \dot{W} \,. \tag{6}$$

and for the SHE (8)–(4) we introduce z^* solving the linear system

$$z_t = \Delta z + \dot{W} \,. \tag{7}$$

Using these stationary processes $z^{*\nu}$ and z^*, we transform the SPDEs to random differential equations and show that this leads to the exact random invariant manifold of the SPDEs (Theorem 6). Such a transformation is frequently invoked in research on SPDEs [16, 17, e.g.]; we verify rigorously the effectiveness of this transformation.

One big difficulty in approximating the random invariant manifolds of the SWE (1)–(2) by that of the SHE (8)–(4), is that second order derivatives in time of u^ν and u cannot be treated path-wise in the usual phase space. The difficulty for u^ν can be overcame by the introduction of $z^{*\nu}$. However, because z^*_t cannot be treated as continuous process, we cannot overcome this difficulty for u by this transformation. Fortunately, by Rohlin's classification of homomorphisms on Lebesgue space (Appendix B), as the distribution of $z^*(\theta_t\omega)$ is the same as that of $z^{*\nu}(\theta_t\omega)$ (Appendix A), there is a measure preserving mapping ψ^ν (Appendix B) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$z^*(\psi^\nu\theta_t\omega) = z^{*\nu}(\theta_t\omega).$$

So we can consider $z^*(\psi^\nu\theta_t\omega)$ instead of $z^*(\theta_t\omega)$. Our result (Theorem 11) on random invariant manifolds implies that the approximate system is

$$\tilde{u}^{\nu}_t(t,x) = \Delta\tilde{u}^{\nu}(t,x) + f(\tilde{u}^{\nu}(t,x)) + \sigma W^{\nu}_t(t,x), \tag{8}$$

where the Wiener process $W^\nu(t,x) = \psi^\nu W(t,x)$. This approximating result also shows that, different from the Smolukowski–Kramers approximation on finite time interval, as $t \to \infty$, for small $\nu > 0$ and almost all $\omega \in \Omega$, the solution $u^\nu(t,x,\omega)$ to SWE (1) is approximated by $u(t,x,\psi^\nu\omega)$, the solution to SHE (8) on the $\psi^\nu\omega$ path. Such transitions of the random parameter ω also appears in approximations of the random invariant manifold for slow-fast stochastic system [27]. However, the transition of ψ^ν here is difficult to be defined explicitly. This is left for future research.

Similar to the analysis of deterministic wave equations [9], we here introduce the change of variables (22) and a new inner product on the phase space (section 4). Because of this change of variables, we restrict the nonlinearity to satisfy $f(0) = 0$, which was also needed for the analysis of deterministic wave equations [9]. Our results generalise the deterministic results [9].

There has been some research on the approximation of the SWE (1)–(2) as $\nu \to 0$ on finite time intervals [7, 8]. But there has been little research on

the approximation of the long time behaviour of the SWE (1)–(2). However, recent research gave an approximation for the long time behaviour in an almost sure sense [18] and distribution [24] in the special case $\sigma = \sqrt{\nu}$.

2 Preliminary

Denote by $L^2(D)$ the set of square integrable functions on $(0, \pi)$, and denote by $\langle \cdot, \cdot \rangle$ the usual inner product, $\| \cdot \|$ the norm on $L^2(D)$. We also denote by $H_0^1(D)$ the usual Sobolev space $W_0^{1,2}(D)$ [1].

Let $A = \Delta$ with zero Dirichlet boundary condition on $(0, \pi)$. Then the operator A generates a strongly continuous semigroup e^{At}, $t \geq 0$, on $L^2(D)$. Denote the eigenvalues of $-A$ by $\lambda_k = k^2$, $k = 1, 2, \ldots$, and the corresponding eigenfunctions $\{e_k\}$ which forms a standard orthogonal basis in $L^2(D)$. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L_f, and then there is a constant $C > 0$ such that

$$|f(\xi)| \leq C(1 + |\xi|) \quad \text{for any} \quad \xi \in \mathbb{R}. \tag{9}$$

Furthermore, we assume

$$L_F \leq \sqrt{\lambda_1}. \tag{10}$$

The above condition ensures the existence of a unique stationary solution to stochastic wave equations (1)–(2) [3].

The Wiener process $\{W(t, x)\}_{t \in \mathbb{R}}$ is assumed to be a two sided, $L^2(D)$-valued, Q-Wiener process with covariance operator Q satisfying

$$\operatorname{Tr} Q < \infty. \tag{11}$$

For our purpose we assume the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ be the canonical probability space with Wiener measure $\mathbb{P}$ [2]. To be more precise, W is the identity on Ω, with

$$\Omega = \left\{ w \in C(\mathbb{R}, L^2(D)) : w(0) = 0 \right\}.$$

Let $\theta_t : (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P}) \to (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ be a metric dynamical system (driven system), that is,

- $\theta_0 = \mathrm{id}$,
- $\theta_t \theta_s = \theta_{t+s}$ for all s, $t \in \mathbb{R}$,
- the map $(t, \omega) \mapsto \theta_t \omega$ is measurable and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

On this canonical probability space Ω, we choose θ_t to be the Wiener shift [2]

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \quad \omega \in \Omega_0, \tag{12}$$

which preserves the Wiener measure $\mathbb{P}$ on Ω. Furthermore, θ_t is ergodic under Wiener measure $\mathbb{P}$. Write $W(t,x)$ as $W(t,x,\omega)$ to show the dependence on $\omega \in \Omega$, then

$$W(\cdot, x, \theta_t\omega) = W(\cdot + t, x, \omega) - W(t, x, \omega).$$

In this view, the stochastic wave equation (1)–(2) is driven by θ_t.

3 Random invariant manifold for stochastic evolutionary equation

Random invariant manifold theory for stochastic evolutionary equations (SEEs) has been developed in lots of research [4, 6, 12, 13, 5, 2, e.g.]. Here we just recall some basic concepts and results.

Let H be a separable Hilbert space with norm $\|\cdot\|_H$ and inner product $\langle\cdot,\cdot\rangle_H$. We consider a stochastic process $\{\varphi(t)\}_{t\geq 0}$ defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$

Definition 1. *A stochastic process $\{\varphi(t)\}_{t\geq 0}$ is called a random dynamical system (*RDS*) over metric dynamical system $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P}, \{\theta_t\}_t)$ if φ is $(\mathcal{B}[0,\infty) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable*

$$\begin{aligned} \varphi : \mathbb{R}^+ \times \Omega \times H &\to H \\ (t,\omega,x) &\mapsto \varphi(t,\omega,x) \end{aligned}$$

and for almost all $\omega \in \Omega$

- *$\varphi(0,\omega) = id$ (on H);*
- *$\varphi(t+s,\omega,x) = \varphi(t,\theta_s\omega,\varphi(s,\omega,x))$ for all $t,s \in \mathbb{R}^+$, $x \in H$ (cocycle property).*

If $\varphi(t,\omega,\cdot) : H \to H$ is continuous, $\{\varphi(t)\}_{t\geq 0}$ is called a continuous RDS.

Definition 2. *A random set $M(\omega)$ is called invariant for* RDS *φ if*

$$\varphi(t,\omega,M(\omega)) \subset M(\theta_t\omega), \quad \textit{for any } t \geq 0\,.$$

If an invariant set $M(\omega)$ is represented by a Lipschitz or C^k mapping $h(\cdot,\omega) : H_1 \to H_2$ with $H = H_1 \oplus H_2$ such that $M(\omega) = \{\xi + h(\xi,\omega) : \xi \in H_1\}$, then we call $M(\omega)$ a Lipschitz or C^k invariant manifold. Furthermore, if H_1 is finite dimensional and $M(\omega)$ attracts exponentially all the orbits of φ, then we call $M(\omega)$ a random stochastic inertial manifold of φ.

For our purpose we consider the RDS defined by the following abstract evolutionary equation with additive noise

$$u_t = Au + F(u) + \sigma\dot{W}\,, \quad u(0) = u_0 \in H\,. \tag{13}$$

Here $F : H \to H$ is globally Lipschitz continuous with Lipschitz constant L_F, W is an H valued Wiener process with trace class covariation operator on H. Furthermore $A : D(A) \subset H \to H$ is a linear operator which generates a strongly continuous semigroup $\{e^{At}\}_{t\geq 0}$ on H, which can be extended to a group $\{e^{At}\}_{t\in\mathbb{R}}$ on H. We assume the following exponential dichotomy.

Condition 3. *With exponents $\beta < \alpha < 0$, and bound $K > 0$, there exists a continuous projection P on H such that*

1. $Pe^{At} = e^{At}P, \quad t \in \mathbb{R}$.
2. *the restriction $e^{At}|_{R(P)}, t \geq 0$, is an isomorphism of the range $R(P)$ of P onto itself, and we denote e^{At} for $t < 0$ the inverse map;*
3. - $\|e^{At}Px\|_H \leq Ke^{\alpha t}\|x\|_H$, $t \leq 0$, *and*
   - $\|e^{At}(I - P)x\|_H \leq Ke^{\beta t}\|x\|_H$, $t \geq 0$.

By the assumption we have the following theorem.

Theorem 4. *Assume Condition 3, then the* SEE *(13) has a unique stationary solution $u^*(t, \omega) = u^*(\theta_t\omega)$.*

Remark 1. There has been lots of research on the existence and uniqueness of stationary solution to stochastic evolutionary equations [10, 11, 15, e.g.]. Here the assumption on $\alpha < 0$ is not essential; for $\alpha > 0$ the above theorem also holds provided the Lipschitz constant L_F is small enough [14].

Suppose $u^*(t, \omega) = u^*(\theta_t\omega)$ is a stationary solution of the SEE (13). We construct a random invariant manifold to the stationary solution u^*. To do this we transform the SEE (13) to a random dynamical system [13]. Define $U = u - u^*$, then

$$U_t = AU + F(u) - F(u^*(\theta_t\omega)), \quad U(0) = U_0 \in H\,. \tag{14}$$

Notice the above system has a unique stationary solution $U = 0$. For any $U_0 \in H$, there is a unique solution $\Phi(t, \omega)U_0$ to equation (14) and $\{\Phi(t, \omega)\}_{t\geq 0}$ defines a continuous random dynamical system on H [13, e.g.]. Then by the Lyapunov–Perron method for random evolutionary equations [13], we have the following theorem.

Theorem 5. *Choose $\eta < 0$ such that spectral gap condition*

$$KL_F\left(\frac{1}{\alpha - \eta} + \frac{1}{\eta - \beta}\right) < 1\,,$$

holds, then there exists a Lipschitz random invariant manifold for SEE *(13), which is given by*

$$M(\omega) = \{(\xi, h(\xi, \omega)) + u^*(\omega) : \xi \in PH\},$$

where $h : PH \to QH$ is a Lipschitz continuous mapping with Lipschitz constant L_h and $h(0) = 0$. Moreover, if

$$KL_F\left(\frac{1}{\alpha-\eta}+\frac{1}{\eta-\beta}\right)+K^2L_hL_F\frac{1}{\alpha-\eta}<1\,, \tag{15}$$

then $M(\omega)$ is a random inertial manifold for the SEE *(13). Furthermore, if $F \in C^1(H,H)$, then the random invariant manifold is also C^1, that is, $h \in C^1(PH,QH)$.*

The Lyapunov–Perron method gives an expression of $h : PH \to QH$ as $h(\xi,\omega) = Q\bar{U}(0,\xi)$ for $\xi \in PH$ with $\bar{U}$ being the unique solution of the following integral equation

$$\begin{aligned}\bar{U}(t,\xi) &= e^{At}(\xi - Pu^*) + \int_0^t e^{A(t-s)}P[F(\bar{u}(s)) - F(u^*(s))]\,ds \\ &\quad + \int_{-\infty}^t e^{A(t-s)}Q[F(\bar{u}(s)) - F(u^*(s))]\,ds \end{aligned}\tag{16}$$

in the Banach space

$$C^-_{\eta,H} = \left\{u \in C((-\infty,0];H) : \sup_{t\le 0} e^{-\eta t}\|u(t)\| < \infty\right\} \tag{17}$$

with norm

$$|u|_{C^-_{\eta,H}} = \sup_{t\le 0} e^{-\eta t}\|u(t)\|\,.$$

However, directly constructing an explicit expression to a stationary solution for a nonlinear SPDE is very difficult. So we use another transformation. Define the stationary process $z^*(t,\omega) = z^*(\theta_t\omega)$ that solves the linear SPDE

$$z_t = Az + \dot{W}\,. \tag{18}$$

Then $z^*(\omega) = \int_{-\infty}^0 e^{-As}dW(s,\omega)$ and

$$z^*(t,\omega) = e^{At}\int_{-\infty}^0 e^{-As}dW(s) + \int_0^t e^{A(t-s)}dW(s). \tag{19}$$

Introduce $V = u - z^*$, then

$$V_t = AV + F(V + z^*(\theta_t\omega)), \quad V(0) = V_0 \in H\,. \tag{20}$$

Then $V^* = u^* - z^*$ is the unique stationary solution to equation (20). Similarly for any $V_0 \in H$, there is a unique solution $\Psi(t,\omega)V_0$ to equation (20) and $\{\Psi(t,\omega)\}_{t\ge 0}$ defines a continuous random dynamical systems on H. By the Lyapunov–Perron method [13], we also have a random invariant manifold $\widetilde{M}(\omega)$, and then $\widetilde{M}(\omega) + z^*(\omega)$ is a random invariant manifold for the SEE (13). The following theorem establishes that this random invariant manifold coincides with $M(\omega)$ in Theorem 5.

Theorem 6.
$$M(\omega) = \widetilde{M}(\omega) + z^*(\omega).$$

Proof. By the Lyapunov–Perron method for a random dynamical system [13], the random dynamical system $\Psi(t, \omega)$ defined by the random evolutionary equation (20) has a random invariant manifold $\widetilde{M}(\omega) = \{(\xi - Pz^*(0), \tilde{h}(\xi, \omega)) : \xi \in PH\}$ where $\tilde{h}(\xi, \omega) = Q\tilde{V}(0, \xi)$ and where $\tilde{V}$ is the unique solution of the integral equation

$$\begin{aligned}\tilde{V}(t, \xi) &= e^{At}(\xi - Pz^*(0)) + \int_0^t e^{A(t-s)} PF(\tilde{V}(s) + z^*(s))\, ds \\ &\quad + \int_{-\infty}^t e^{A(t-s)} QF(\tilde{V}(s) + z^*(s))\, ds\end{aligned}$$

in space $C^-_{\eta,H}$. Since the stationary solution $V^*(\omega)$ lies on this random invariant manifold, choosing $\xi = Pu^*(0)$, we have $\tilde{V}(t, Pu^*(0)) = V^*(t, \omega)$. Then by the expression for z^*,

$$\begin{aligned}u^*(t, \omega) &= \tilde{V}(t, Pu^*(0)) + z^*(t) \\ &= e^{At} Pu^*(\omega) + \int_0^t e^{A(t-s)} PF(u^*(s))\, ds + \int_0^t e^{A(t-s)} P dW(s) \\ &\quad + \int_{-\infty}^t e^{A(t-s)} QF(u^*(s))\, ds + \int_{-\infty}^t e^{A(t-s)} Q dW(s)\end{aligned}$$

and

$$u^*(\omega) = u^*(0, \omega) = Pu^*(\omega) + \int_{-\infty}^0 e^{-As} QF(u^*(s))\, ds + \int_{-\infty}^0 e^{-As} Q dW(s). \tag{21}$$

Notice that by the integral equation (16), the solution $\bar{u}$ to the SEE (13) with initial value $\bar{u}(0) = (\xi, h(\xi, \omega)) + u^*(\omega)$ is

$$\begin{aligned}\bar{u}(t) &= \bar{U}(t) + u^*(t) \\ &= e^{At}\xi + \int_0^t e^{A(t-s)} PF(\bar{u}(s))\, ds + \int_{-\infty}^t e^{A(t-s)} QF(\bar{u}(s))\, ds + u^*(t) \\ &\quad - e^{At} Pu^* - \int_0^t e^{A(t-s)} PF(u^*(s))\, ds - \int_{-\infty}^t e^{A(t-s)} QF(u^*(s))\, ds\,.\end{aligned}$$

Rewrite the last three terms in the above equality and by (21)

$$
\begin{aligned}
& e^{At}\left[Pu^* + \int_{-\infty}^{0} e^{-As} QF(u^*(s))\,ds + \int_{-\infty}^{0} e^{-As} QdW(s)\right] \\
& + \int_0^t e^{A(t-s)} F(u^*(s))\,ds + \int_0^t e^{A(t-s)} dW(s) \\
& - \int_{-\infty}^{t} e^{A(t-s)} QdW(s) - \int_0^t e^{A(t-s)} PdW(s) \\
= {} & u^*(t,\omega) - \int_{-\infty}^{t} e^{A(t-s)} QdW(s) - \int_0^t e^{A(t-s)} PdW(s).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\bar{u}(t) &= e^{At}\xi + \int_0^t e^{A(t-s)} PF(\bar{u}(s))\,ds + \int_{-\infty}^{t} e^{A(t-s)} QF(\bar{u}(s))\,ds \\
&\quad + \int_{-\infty}^{t} e^{A(t-s)} QdW(s) + \int_0^t e^{A(t-s)} PdW(s) \\
&= \tilde{V}(t,\xi) + z^*(t).
\end{aligned}
$$

The proof is complete. □

The above theorem shows that if the SEE (13) has a unique stationary solution u^*, then the random invariant manifold $\mathcal{M}(\omega)$ to the stationary solution u^* can be derived from the random invariant manifold $\widetilde{\mathcal{M}}(\omega)$ to V^*, the stationary solution of (20), by the transformation $V = u - z^*$.

4 Random invariant manifold for SWEs

We construct a random inertial invariant manifold for the SWE (1)–(2) with fixed parameter $\nu > 0$.

By the result of Wang and Lv [24], there is a stationary solution $(u^{*\nu}, u_t^{*\nu}) \in H_0^1(D) \times L^2(D)$. Furthermore, this stationary solution is unique provided the Lipschitz inequality (10) holds [3]. By the discussion at the end of the last section, we use the transformation $\bar{u}^\nu = u^\nu - z^{*\nu}$, and for technical reasons we make the change of variables

$$
\bar{u}_t^\nu = -\frac{1}{2\nu}\bar{u}^\nu + \frac{1}{\nu}\bar{v}^\nu \quad \text{and} \quad \bar{U}^\nu = (\bar{u}^\nu, \bar{v}^\nu). \tag{22}
$$

The above change of variables is similar to that for the deterministic wave equation [9] and to that in previous research on stochastic wave equations [16, 17]. For convenience, we give a simple description of the construction of the random inertial manifold. By the definition of $\bar{U}^\nu$ we have a random differential equation

$$
\bar{U}_t^\nu(t,\omega) = C\bar{U}^\nu(t,\omega) + F(\bar{U}^\nu(t,\omega), \theta_t\omega) \tag{23}
$$

where

$$C = \begin{bmatrix} -\frac{1}{2\nu} & \frac{1}{\nu} \\ \frac{1}{4\nu} + A & -\frac{1}{2\nu} \end{bmatrix}, \quad F(\bar{U}^\nu, \omega) = \begin{bmatrix} 0 \\ f(\bar{u}^\nu + z^{*\nu}) \end{bmatrix}.$$

We apply Theorem 5 and Theorem 6 to construct a random invariant manifold for equation (23) based upon a stationary solution $(u^{*\nu}, u_t^{*\nu})$.

We first state some facts on the linear operator C. Let $E = H_0^1(D) \times L^2(D)$ and $N > 0$ be an integer. Set

$$\begin{aligned} E_{11} &= \operatorname{span}\left\{ \begin{bmatrix} e_k \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_k \end{bmatrix} : k = 1, \dots, N \right\} \\ E_{22} &= \operatorname{span}\left\{ \begin{bmatrix} e_k \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_k \end{bmatrix} : k = N+1, N+2, \dots \right\}. \end{aligned}$$

It is evident that $E = E_{11} \oplus E_{22}$, that E_{11} is orthogonal to E_{22} by the orthogonality of $\{e_k\}$, and that $\dim E_{11} = 2N$. Moreover, both E_{11} and E_{22} are invariant subspaces of the operator C.

Since the eigenvalues of A are $-k^2$ with corresponding eigenvectors e_k, $k = 1, 2, \dots$, by restricting C to E_{11}, the eigenvalues of $C|_{E_{11}}$ are

$$\lambda_k^\pm = \frac{-1 \pm \sqrt{1 - 4\nu k^2}}{2\nu}, \quad k = 1, 2, \dots, N,$$

with corresponding eigenvectors

$$e_k^\pm = \begin{bmatrix} e_k \\ \pm\frac{\sqrt{1-4\nu k^2}}{2} e_k \end{bmatrix}, \quad k = 1, 2, \dots, N.$$

Let

$$E_1 = \operatorname{span}\{e_k^+ : k = 1, \dots, N\}, \quad E_{-1} = \operatorname{span}\{e_k^- : k = 1, \dots, N\}.$$

By this definition $E_{11} = E_1 \oplus E_{-1}$, and E_1 and E_{-1} are invariant subspaces of the operator C. Let P_1 and P_{-1} be the corresponding spectral projections [20] and P_{22} be the unique orthogonal projection onto E_{22}. Then there exist a decomposition $E = E_1 \oplus E_{-1} \oplus E_{22}$ with projections P_1, P_{-1}, P_{22} respectively. Note that E_1 is not orthogonal to E_{-1}. To overcome this we invoke an equivalent inner product on E, as defined for the deterministic wave equations [19], to ensure E_1 is orthogonal to E_{-1}.

Let $U_i = (u_i, v_i)$, $i = 1, 2$, be two elements of E or E_{11}, E_{22}. Assume $\frac{1}{2\sqrt{\nu}} > N + 1$, and define the new inner products on E_{11} and E_{22} as

$$\begin{aligned} \langle U_1, U_2 \rangle_{E_{11}} &= \langle \nu A u_1, u_2 \rangle + \frac{1}{4}\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle, \\ \langle U_1, U_2 \rangle_{E_{22}} &= \langle -A u_1, u_2 \rangle + \left(\frac{1}{4\nu} - 2(N+1)^2 \right) \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product of $L^2(D)$. Define the new inner product on E by

$$\langle U, V \rangle_E = \langle U_{11}, V_{11} \rangle_{E_{11}} + \langle U_{22}, V_{22} \rangle_{E_{22}}$$

where $U = U_{11} + U_{22}$ and $V = V_{11} + V_{22}$ with $U_{ii}, V_{ii} \in E_{ii}$, $i = 1, 2$. The corresponding norm is denoted by $\| \cdot \|_E$.

Since $\frac{1}{2\sqrt{\nu}} > N + 1$, $\langle \cdot, \cdot \rangle_{E_{11}}$ is equivalent to the usual inner product on E_{11}, and $\langle \cdot, \cdot \rangle_{E_{22}}$ is equivalent to the usual inner product on E_{22}. Hence the new inner product $\langle \cdot, \cdot \rangle_E$ is equivalent to the usual inner product on E [19].

In terms of this new inner product, by the orthogonality of $\sin kx$, direct methods verify that $E_{-1} \perp E_{22}$ and $E_1 \perp E_{22}$. Moreover, $E_1 \perp E_{-1}$. Let $E_2 = E_{-1} \oplus E_{22}$, then $E_1 \perp E_2$.

By the definition of the new inner product, for $U = (0, v) \in E$,

$$\|U\|_E = \|v\|_{L^2(D)}, \tag{24}$$

and for any $U = (u, v) \in E$,

$$\|U\|_E \geq \sqrt{\frac{1}{4} - \nu(N+1)^2} \|u\|_{L^2(D)} \, . \tag{25}$$

Let C_1, C_2, C_{-1}, C_{22} denote $C|_{E_1}, C|_{E_2}, C|_{E_{-1}}, C|_{E_{22}}$, respectively. Then similar to Mora's bounds [19],

$$\|e^{C_1 t}\| \leq e^{\lambda_N^+ t} \quad \text{for } t \leq 0\,, \tag{26}$$

$$\|e^{C_{-1} t}\| \leq e^{\lambda_N^- t} \quad \text{for } t \geq 0\,, \tag{27}$$

$$\|e^{C_{22} t}\| \leq e^{\lambda_{N+1}^+ t} \quad \text{for } t \geq 0\,. \tag{28}$$

By the bounds (27) and (28), we have

$$\|e^{C_2 t}\| \leq e^{\lambda_{N+1}^+ t} \quad \text{for } t \geq 0\,.$$

For the nonlinearity F, in terms of the new norm, by (24) and (25),

$$\begin{aligned} \|F(\bar{U}_1, \omega) - F(\bar{U}_2, \omega)\|_E &= \|f(\bar{u}_1 + z^{*\nu}) - f(\bar{u}_2 + z^{*\nu})\|_{L^2(D)} \\ &\leq L_f \|\bar{u}_1 - \bar{u}_2\|_{L^2(D)} \\ &\leq \frac{L_f}{\sqrt{\frac{1}{4} - \nu(N+1)^2}} \|\bar{U}_1 - \bar{U}_2\|_E \\ &\leq 3L_f \|\bar{U}_1 - \bar{U}_2\|_E. \end{aligned}$$

So F is Lipschitz with respect to $\bar{U}$ and the Lipschitz constant is independent of ν provided the parameter ν is small.

Notice that by choosing $\alpha = \lambda_N^+$, $\beta = \lambda_{N+1}^+$ and $\eta = (\lambda_N^+ + \lambda_{N+1}^+)/2$, for $\nu > 0$ small enough, the gap condition (15) in Theorem 5 holds. Then a similar discussion to that by Liu [16] and Lu & Schmalfuß [17] leads to the following theorem.

Theorem 7. *There exists $\nu_0 > 0$ such that for any $\nu \in (0, \nu_0)$, there is an N-dimensional inertial manifold $\bar{\mathcal{M}}_E^\nu(\omega)$ for equation (23), which is represented by*

$$\bar{\mathcal{M}}_E^\nu(\omega) = \{(\xi, h^\nu(\xi, \omega)) : \xi \in E_1\}$$

with

$$h^\nu(\cdot, \omega) : E_1 \to E_2$$

being Lipschitz continuous. Moreover, if $f \in C^1(L^2(D), L^2(D))$, then the random invariant manifold is C^1, that is $h^\nu \in C^1(E_1, E_2)$.

For our purposes we need some estimates of the solution on the random invariant manifold $\bar{\mathcal{M}}_E^\nu(\omega)$. For $\bar{U}_0^\nu = (\xi, h^\nu(\xi, \omega)) \in \bar{\mathcal{M}}_E^\nu(\omega)$, by the invariance of $\bar{\mathcal{M}}_E^\nu(\omega)$, $\bar{U}^\nu(t, \omega)$, the solution of (23) with $\bar{U}^\nu(0) = \bar{U}_0^\nu$, lies on $\bar{\mathcal{M}}_E^\nu(\theta_t \omega)$. Then, by the construction of the random invariant manifold and the uniqueness of solutions, for $t \leq 0$

$$\begin{aligned} \bar{U}^\nu(t, \omega) &= e^{Ct}\xi + \int_0^t P_1 e^{C(t-s)} F(\bar{U}^\nu(s, \omega), \omega)\, ds \\ &\quad + \int_{-\infty}^t (P_{-1} + P_{22}) e^{C(t-s)} F(\bar{U}^\nu(s, \omega), \omega)\, ds\,. \end{aligned}$$

Notice that

$$\|F(\bar{U}^\nu)\|_E \leq \|f(\bar{u}^\nu + z^{*\nu})\|_{L^2(D)} \leq 3L_f(\|\bar{U}^\nu\|_E + \|z^{*\nu}\| + 1).$$

Then, by the gap condition (15), a direct calculation yields

$$|\bar{U}^\nu|_{C^-_{\eta,E}} \leq R_1(\omega) \tag{29}$$

for some tempered random variable R_1. Here the Banach space $C^-_{\eta,E}$ is defined by (17) through replacing H by E. Next we need the same estimate on $\bar{U}_t^\nu$. Since $\bar{U}^\nu(t, \omega)$ lies on $\bar{\mathcal{M}}^\nu(\omega)$, we have

$$\bar{U}^\nu(t, \omega) = \bar{U}_N^\nu(t, \omega) + h^\nu(\bar{U}_N^\nu(t, \omega), \omega)$$

with $\bar{U}_N^\nu(0, \omega) = \xi$ and $\bar{U}_N^\nu(t, \omega) \in P_1 E$ for $t \in \mathbb{R}$. Moreover,

$$\dot{\bar{U}}_N^\nu(t, \omega) = C_1 \bar{U}_N^\nu(t, \omega) + P_1 F(\bar{U}_N^\nu(t, \omega) + h^\nu(\bar{U}_N^\nu(t, \omega), \omega)).$$

Then by (29), (26), the spectrum gap condition (15) and the Lipschitz property of h^ν,

$$|\dot{\bar{U}}_N^\nu(t, \omega)|_{C^-_{\eta,E}} \leq R_2'(\omega). \tag{30}$$

Notice that

$$\dot{\bar{U}}^{\nu}(t,\omega) = \dot{\bar{U}}^{\nu}_{N}(t,\omega) + Dh^{\nu}(\bar{U}^{\nu}_{N}(t,\omega),\omega)\dot{\bar{U}}^{\nu}_{N}(t,\omega),$$

then by the bound (30), for some tempered random variable R_2,

$$|\dot{\bar{U}}^{\nu}(t,\omega)|_{C^{-}_{\eta,E}} \le R_2(\omega). \tag{31}$$

5 Approximation of random inertial manifold

This section addresses the approximation of $\bar{\mathcal{M}}^{\nu}_{E}$ for small $\nu > 0$. First we consider the stochastic heat equation (8)–(4). Recall the stationary process z^* that solves (7). We make the transformation $\tilde{u} = u - z^*$, and derive $\tilde{u}$ satisfies the RDS

$$\tilde{u}_t(t,\omega) = \Delta\tilde{u}(t,\omega) + f(\tilde{u}(t,\omega) + z^*(\theta_t\omega)). \tag{32}$$

Notice that under our assumptions, the stochastic nonlinear heat equation (8)–(4) has a unique stationary solution. By Theorem 5 the following theorem holds.

Theorem 8. *Assume $f \in C^1(L^2(D), L^2(D))$ and $N > 0$ large enough Then the random equation (32) has an N-dimensional C^1 random inertial manifold $\widetilde{\mathcal{M}}_{L^2(D)}(\omega)$ with*

$$\widetilde{\mathcal{M}}_{L^2(D)}(\omega) = \big\{(\zeta, h(\zeta,\omega)) : \zeta \in P_N L^2(D)\big\}$$

where P_N is the orthogonal projection from $L^2(D)$ to the N-dimensional space $\operatorname{span}\{e_1, e_2, \dots, e_N\}$.

Now we define the following random set in E

$$\widetilde{\mathcal{M}}_{E}(\omega) = \Big\{(\tilde{u}_0, \tilde{u}_t(0,\tilde{u}_0)) : \tilde{u}_0 \in \widetilde{\mathcal{M}}_{L^2(D)}(\omega)\Big\}, \tag{33}$$

and bounded random set

$$\widetilde{\mathcal{M}}_{ER}(\omega) = \Big\{(\tilde{u}_0, \tilde{u}_t(0,\tilde{u}_0)) : \tilde{u}_0 = \zeta + h(\zeta,\omega) \in \widetilde{\mathcal{M}}_{L^2(D)}(\omega), \|\zeta\|_{L^2(D)} < R\Big\}, \tag{34}$$

where $\tilde{u}(t,\tilde{u}_0)$ is the unique solution of the RDS (32) with initial value $\tilde{u}(0,\tilde{u}_0) = \tilde{u}_0$, and where $R > 0$ is an arbitrary constant.

As mentioned in the Introduction, we have to consider the RDS (32) on $\psi^{\nu}\omega$. For this we define $\tilde{u}^{\nu}(t,\omega)$ on the fiber $\psi^{\nu}\omega$ solving

$$\tilde{u}^{\nu}_t(t,\omega) = \Delta\tilde{u}^{\nu}(t,\omega) + f(\tilde{u}^{\nu}(t,\omega) + z^*(\psi^{\nu}\theta_t\omega)). \tag{35}$$

Then by the Lyapunov–Perron method we have an N-dimensional random invariant manifold which is exactly $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^{\nu}\omega)$. We give a relation between $\bar{M}^{\nu}_{E}(\omega)$ and $\widetilde{\mathcal{M}}_{ER}(\psi^{\nu}\omega)$.

We need some estimates of the solution of equation (35) on $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\omega)$. For $\tilde{u}_0 = (\zeta, h(\zeta,\psi^\nu\omega)) \in \widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\omega)$, then $\tilde{u}^\nu(t,\omega)$, the solution to equation (35) with $\tilde{u}^\nu(0,\omega) = \tilde{u}_0$, by the invariance of $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\omega)$, lies on $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\theta_t\omega)$. Then, by a similar discussion for $\bar{U}^\nu$, for $t \le 0$

$$\begin{aligned}\tilde{u}^\nu(t,\omega) &= e^{At}\zeta + \int_0^t P_N e^{A(t-s)} f(\tilde{u}^\nu(s,\omega) + z^{*\nu}(s,\omega))\,ds \\ &\quad + \int_{-\infty}^t (\mathrm{Id} - P_N) e^{A(t-s)} f(\tilde{u}^\nu(s,\omega) + z^{*\nu}(s,\omega))\,ds\,.\end{aligned}$$

By the gap condition and the Lipschitz property of f, for some tempered random variable R_3,

$$|\tilde{u}^\nu|_{C^-_{\eta,L^2(D)}} \le R_3(\omega).$$

Further, we need the following estimate of $\tilde{u}^\nu_{tt}$ with $\tilde{u}^\nu(t,\omega)$ lying on the random invariant manifold $\widetilde{\mathcal{M}}_{ER}(\psi^\nu\theta_t\omega)$.

Lemma 9. *Assume the conditions of Theorem 8. For each $R > 0$, such that for $\|\zeta\|_{L^2(D)} \le R$ and $\zeta \in P_N L^2(D)$, then almost surely*

$$\nu\|e^{-\eta t}\tilde{u}^\nu_{tt}(t, \zeta + h(\zeta,\psi^\nu\omega),\omega)\|_{L^2(D)} \to 0\,, \quad t \le 0\,,$$

where $\tilde{u}^\nu$ is the unique solution of the RDS *(32) with $\tilde{u}^\nu(0) = \zeta + h(\zeta,\psi^\nu\omega)$.*

Proof. By the invariance of $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\omega)$ for $\tilde{u}_0 = \zeta + h(\zeta,\psi^\nu\omega)$

$$\tilde{u}^\nu(t,\tilde{u}_0,\omega) = \tilde{u}^\nu_N(t,\zeta,\omega) + h(\tilde{u}^\nu_N(t,\zeta,\omega),\psi^\nu\omega)$$

with

$$\tilde{u}^\nu_{N,t} = \Delta\tilde{u}^\nu_N + P_N f(\tilde{u}^\nu_N + h(\tilde{u}^\nu_N,\psi^\nu\omega) + z^{*\nu}(t,\omega)), \quad \tilde{u}^\nu_N(0) = \zeta\,. \tag{36}$$

Here we use the equality $z^*(\psi^\nu\theta_t\omega) = z^{*\nu}(\theta_t\omega) = z^{*\nu}(t,\omega)$. By a similar discussion to that for the Lyapunov–Perron method to construct a random invariant manifold, we can construct a unique solution $\tilde{u}_N$ to (36) which is in the space $C^-_{\eta,P_N L^2(D)}$. Then

$$\|e^{-\eta t}\tilde{u}^\nu_N(t,\omega)\|_{L^2(D)} \le C(\omega), \quad t \le 0\,,$$

for some random constant $C(\omega)$ which is independent of parameter ν. By Theorem 8, $h \in C^1$ and Lipschitz, we have

$$\nu\|\tilde{u}^\nu_t\|_{L^2(D)} = \nu\|\tilde{u}^\nu_{N,t}\|_{L^2(D)} + \nu L_h\|\tilde{u}^\nu_{N,t}\|_{L^2(D)}\,. \tag{37}$$

Then by (36) and above estimate for $e^{-\eta t}\tilde{u}^\nu_N$,

$$\nu\|e^{-\eta t}\tilde{u}^\nu_t(t,\omega)\|_{L^2(D)} \to 0 \quad \text{almost surely for any } t \le 0\,, \quad \text{as} \quad \nu \to 0\,.$$

Now let $\tilde{w}^\nu = \nu\tilde{u}^\nu_t$, then $\tilde{w}^\nu \in C^-_{\eta,L^2(D)}$ and

$$\tilde{w}^\nu_t(t,\omega) = A^\nu(t,\omega)\tilde{w}^\nu(t,\omega) + \nu B^\nu(t,\omega), \quad \tilde{w}^\nu(0) = \nu\tilde{u}^\nu_t(0),$$

with

$$\begin{aligned} A^\nu(t,\omega) &= \Delta + Df(\tilde{u}^\nu(t,\omega) + z^{*\nu}(t,\omega)), \\ B^\nu(t,\omega) &= Df(\tilde{u}^\nu(t,\omega) + z^{*\nu}(t,\omega))z^{*\nu}_t(t,\omega). \end{aligned}$$

Under the assumptions of Theorem 8, $\tilde{w}^\nu \in C^-_{\eta,L^2(D)}$ is equivalent to the statement that $\tilde{w}^\nu$ has the following form for $t \le 0$,

$$\begin{aligned} \tilde{w}^\nu(t,\omega) &= S^\nu(t,\omega)P_N\tilde{w}^\nu(0) + \nu\int_0^t S^\nu(t-s,\omega)P_N B^\nu(s,\omega)\,ds \\ &\quad + \nu\int_{-\infty}^t S^\nu(t-s,\psi^\nu\omega)(I-P_N)B^\nu(s,\omega)\,ds \end{aligned}$$

where

$$S^\nu(t,\omega) = \exp\left\{\int_0^t A^\nu(s,\omega)\,ds\right\}.$$

Notice that $\tilde{u}^\nu(t,\omega)$ lies on the random invariant manifold $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\theta_t\omega)$, that $z^{*\nu}$ is stationary, that by the assumption (10) $S^\nu(t,\omega)$ is nonuniformly pseudo-hyperbolic [6], and that $B^\nu(t,\omega)$ is tempered and locally integrable in t. Then by a similar discussion to that for random evolutionary equation [6], we also can construct a random invariant manifold. Then, by the same discussion above for the estimates $\nu\|e^{-\eta t}\tilde{u}^\nu_t(t,\omega)\|_{L^2(D)}$ and the estimate (41), the bound on $\nu z^{*\nu}_t(t,\omega)$ in Appendix A, we have the estimate

$$\nu\|e^{-\eta t}\tilde{w}^\nu_t(t,\omega)\|_{L^2(D)} \to 0 \quad \text{almost surely for any } t \le 0\,, \quad \text{as} \quad \nu \to 0\,,$$

which completes the proof. □

Now we establish the following theorem on the relation between $\bar{\mathcal{M}}^\nu_E(\omega)$ and $\widetilde{\mathcal{M}}_{ER}(\psi^\nu\omega)$.

Theorem 10. *Suppose $f \in C^1(L^2(D), L^2(D))$ and $N > 0$ large enough. Then for any $R > 0$*

$$\lim_{\nu\to 0} \operatorname{dist}_E\left(\widetilde{\mathcal{M}}_{ER}(\psi^\nu\omega), \bar{\mathcal{M}}^\nu_E(\omega)\right) = 0\,.$$

Proof. We adapt the discussion for the deterministic case [9].

Let any element $(\tilde{u}_0, \tilde{u}^\nu_t(0,\tilde{u}_0)) \in \widetilde{M}_{ER}(\psi^\nu\omega)$ with $\tilde{u}^\nu$ satisfying equation (35) with initial condition $\tilde{u}^\nu(0) = \tilde{u}_0$. Define

$$\tilde{U}^\nu(t) = \left(\tilde{u}^\nu(t), \frac{1}{2}\tilde{u}^\nu(t) + \nu\tilde{u}^\nu_t(t)\right),$$

then $\tilde{U}^\nu = (\tilde{u}^\nu, \tilde{v}^\nu)$ satisfies

$$\dot{\tilde{U}}^\nu(t,\omega) = C\tilde{U}^\nu(t,\omega) + \begin{bmatrix} 0 \\ f(\tilde{u}^\nu(t,\omega) + z^{*\nu}(t,\omega)) \end{bmatrix} + \begin{bmatrix} 0 \\ \nu \tilde{u}^\nu_{tt}(t,\omega) \end{bmatrix}.$$

Let $\bar{U}^\nu \in \bar{\mathcal{M}}^\nu_E(\omega)$ be a solution of the RDS (23) and $0 < \nu < \nu_0$. Let

$$\hat{U}^\nu(t,\omega) = \tilde{U}^\nu(t,\omega) - \bar{U}^\nu(t,\omega).$$

Hence, $\hat{U}^\nu(t,\omega)$ satisfies

$$\begin{aligned} \dot{\hat{U}}^\nu(t,\omega) &= C\hat{U}^\nu(t,\omega) + \begin{bmatrix} 0 \\ \nu \tilde{u}^\nu_{tt}(t,\omega) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ f(\tilde{u}^\nu(t,\omega) + z^{*\nu}(t,\omega)) - f(\bar{u}^\nu(t,\omega) + z^{*\nu}(t,\omega)) \end{bmatrix}. \end{aligned}$$

Notice that by the Lipschitz property of f and by Lemma 9,

$$\Delta \tilde{u}^\nu(t,\omega) = \tilde{u}^\nu_t(t,\omega) - f(\tilde{u}^\nu(t,\omega) + z^{*\nu}(t,\omega)) \in C^-_{\eta, L^2(D)}.$$

Then by the interpolation between $H^2(D)$ and $L^2(D)$,

$$\tilde{u}^\nu \in C^-_{\eta, H^1_0}.$$

By Lemma 9 we have $\hat{U}^\nu \in C^-_{\eta,E}$, then by the construction of solution in $C^-_{\eta,E}$,

$$\begin{aligned} \hat{U}^\nu(t) &= e^{Ct} P \hat{U}^\nu(0) \\ &\quad + \int_0^t e^{C(t-s)} P \left\{ \begin{bmatrix} 0 \\ f(\tilde{u}^\nu + z^{*\nu}) - f(\bar{u}^\nu + z^{*\nu}) \end{bmatrix} + \begin{bmatrix} 0 \\ \nu \tilde{u}^\nu_{tt} \end{bmatrix} \right\} ds \\ &\quad + \int_{-\infty}^t e^{C(t-s)} Q \left\{ \begin{bmatrix} 0 \\ f(\tilde{u}^\nu + z^{*\nu}) - f(\bar{u}^\nu + z^{*\nu}) \end{bmatrix} + \begin{bmatrix} 0 \\ \nu \tilde{u}^\nu_{tt} \end{bmatrix} \right\} ds. \end{aligned}$$

Since $PE = E_1$ is of finite dimension, we can choose $\bar{u}(0)$ and $\bar{u}_t(0)$ such that $P\hat{U}(0) = 0$. Hence,

$$\begin{aligned} & e^{-\eta t} \|\hat{U}^\nu(t)\|_E \\ & \le e^{-\eta t} \int_t^0 e^{\lambda^+_N (t-s)} \left\{ \left\| \begin{bmatrix} 0 \\ f(\tilde{u}^\nu + z^{*\nu}) - f(\bar{u}^\nu + z^{*\nu}) \end{bmatrix} \right\|_E + \left\| \begin{bmatrix} 0 \\ \nu \tilde{u}^\nu_{tt} \end{bmatrix} \right\|_E \right\} ds \\ & \quad + e^{-\eta t} \int_{-\infty}^t e^{\lambda^+_{N+1}(t-s)} \left\{ \left\| \begin{bmatrix} 0 \\ f(\tilde{u}^\nu + z^{*\nu}) - f(\bar{u}^\nu + z^{*\nu}) \end{bmatrix} \right\|_E + \left\| \begin{bmatrix} 0 \\ \nu \tilde{u}^\nu_{tt} \end{bmatrix} \right\|_E \right\} ds \\ & \le 3L_f \int_t^0 e^{(\lambda^+_N - \eta)(t-s)} \|\hat{U}^\nu\|_{C^-_{\eta,E}} ds + 3L_f \int_{-\infty}^t e^{(\lambda^+_{N+1} - \eta)(t-s)} \|\hat{U}^\nu\|_{C^-_{\eta,E}} ds \\ & \quad + \nu e^{-\eta t} \int_t^0 e^{\lambda^+_N (t-s)} \|\tilde{u}^\nu_{tt}\| ds + \nu e^{-\eta t} \int_{-\infty}^t e^{\lambda^+_{N+1}(t-s)} \|\tilde{u}^\nu_{tt}\| ds. \end{aligned}$$

Then by Lemma 9, for N large enough we have $\|\hat{U}^\nu\|_{C^-_{\eta,E}} \to 0$, as $\nu \to 0$. Hence, $\|\hat{U}^\nu(0)\| \to 0$, as $\nu \to 0$. The proof is complete. $\square$

Next we show the approximation of the random dynamics u^ν. For this we define the random sets

$$\bar{\mathcal{M}}^\nu_{L^2(D)}(\omega) = \{\bar{u} : \bar{U} = (\bar{u}, \bar{v}) \in \bar{\mathcal{M}}^\nu_E(\omega) \quad \text{for some } \bar{v} \in L^2(D)\}$$

and

$$\bar{\mathcal{M}}^\nu_{L^2(D),R}(\omega) = \left\{\bar{u} \in \bar{\mathcal{M}}^\nu_{L^2(D)}(\omega) : \bar{U} = (\bar{u}, \bar{v}) = \xi + h^\nu(\xi, \omega),\ \|\xi\|_{L^2(D)} \le R\right\}.$$

We next prove that for small parameter $\nu > 0$, $\bar{\mathcal{M}}^\nu_{L^2(D),R}(\omega)$ is approximated by $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\omega)$.

Theorem 11. *Suppose $f \in C^1(L^2(D), L^2(D))$ and $N > 0$ large enough. Then for any $R > 0$*

$$\lim_{\nu \to 0} \mathrm{dist}_{L^2(D)}\left(\bar{\mathcal{M}}^\nu_{L^2(D),R}(\omega), \widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\omega)\right) = 0\,.$$

Proof. The proof is similar to that of the deterministic case. Let $\bar{U}_0 = (\bar{u}_0, \bar{v}_0)$ with $\bar{u}_0 \in \bar{\mathcal{M}}^\nu_{L^2(D),R}(\omega)$ and $\bar{U}^\nu(t,\omega) = (\bar{u}^\nu(t,\omega), \bar{v}^\nu(t,\omega))$ be the solution of (23) with $\bar{U}^\nu(0) = \bar{U}(0)$. By the invariance of $\bar{\mathcal{M}}^\nu_E(\omega)$,

$$\bar{u}^\nu_t = A\bar{u}^\nu + f(\bar{u}^\nu + z^{*\nu}) - \nu\bar{u}^\nu_{tt}\,.$$

Let $\tilde{u}^\nu(t,\omega)$ be the solution of equation (35) on the random inertial manifold $\widetilde{\mathcal{M}}_{L^2(D)}(\psi^\nu\omega)$ with $\tilde{u}^\nu(0) = \tilde{u}_0 \in L^2(D)$. Thus $\hat{u}^\nu(t,\omega) = \tilde{u}^\nu(t,\omega) - \bar{u}^\nu(t,\omega)$ satisfies

$$\begin{aligned} \hat{u}^\nu_t(t,\omega) &= A\hat{u}^\nu(t,\omega) + f(\tilde{u}^\nu(t,\omega) + z^{*\nu}(t,\omega)) \\ &\quad - f(\bar{u}^\nu(t,\omega) + z^{*\nu}(t,\omega)) + \nu\bar{u}^\nu_{tt}(t, \theta_{-t}\omega). \end{aligned}$$

Notice that $\tilde{u}^\nu(t,\omega) \in C^-_{\eta,L^2(D)}$ and $\bar{U}^\nu \in \bar{\mathcal{M}}^\nu_E(\omega)$, we have $\hat{u}^\nu \in C^-_{\eta,L^2(D)}$. Then,

$$\begin{aligned} \hat{u}^\nu(t,\omega) &= e^{At}P_N\hat{u}^\nu(0) + \int_0^t e^{A(t-s)}P_N[f(\tilde{u}^\nu(s,\omega) + z^{*\nu}(s,\omega)) \\ &\quad - f(\bar{u}^\nu(s,\omega) + z^{*\nu}(s,\omega)) + \nu\bar{u}^\nu_{tt}(s,\omega)]\,ds \\ &\quad + \int_{-\infty}^t e^{A(t-s)}(I - P_N)[f(\tilde{u}^\nu(s,\omega) + z^{*\nu}(s,\omega)) \\ &\quad - f(\bar{u}^\nu(s,\omega) + z^{*\nu}(s,\omega)) + \nu\bar{u}^\nu_{tt}(s,\omega)]\,ds\,. \end{aligned}$$

We need an estimate on $\nu\bar{u}^\nu_{tt}$. Suppose we have the following expansion in the orthonormal basis of the eigenfunctions e_k of A, $\bar{u}^\nu_t = \sum_{k=1}^\infty \bar{u}^\nu_{t,k}e_k$.

Then by integration by parts,

$$
\begin{aligned}
& \sup_{t\le 0} e^{-\eta t}\left\|\int_0^t e^{A(t-s)}P_N\bar{u}^\nu_{tt}(s)\,ds\right\|_{L^2(D)} \\
= \;& \sup_{t\le 0} e^{-\eta t}\left\|P_N\bar{u}^\nu_t(t)-e^{At}P_N\bar{u}^\nu_t(0)-\sum_{k=1}^N k^2\int_0^t e^{-k^2(t-s)}\bar{u}^\nu_{t,k}(s)e_k\,ds\right\|_{L^2(D)} \\
\le \;& \sup_{t\le 0} e^{-\eta t}\|P_N\bar{u}^\nu_t(t)\|_{L^2(D)}+\sup_{t\le 0} e^{-(\eta+N^2)t}\|P_N\bar{u}^\nu_t(0)\|_{L^2(D)} \\
& +\sup_{t\le 0} e^{-\eta t}\left\|\sum_{k=1}^N k^2\int_0^t e^{-k^2(t-s)}\bar{u}^\nu_{t,k}(s)e_k\,ds\right\|_{L^2(D)}.
\end{aligned}
$$

For the last term in the above equation, we consider its square as

$$
\begin{aligned}
& \sup_{t\le 0} e^{-2\eta t}\left\|\sum_{k=1}^N k^2\int_0^t e^{-k^2(t-s)}\bar{u}^\nu_{t,k}(s)e_k\,ds\right\|^2_{L^2(D)} \\
= \;& \sup_{t\le 0}\sum_{k=1}^N\left[k^2\int_0^t e^{-k^2(t-s)}e^{-\eta(t-s)}e^{-\eta s}\|\bar{u}^\nu_{t,k}(s)\|_{L^2(D)}\,ds\right]^2 \\
\le \;& \sup_{t\le 0}\sum_{k=1}^N\left[k^2\int_0^t e^{-k^2(t-s)}e^{-\eta(t-s)}\,ds\right]^2\|\bar{u}^\nu_{t,k}\|^2_{C_{\eta,\mathbb{R}}} \\
\le \;& \|P_N\bar{u}^\nu_t\|^2_{C^-_{\eta,L^2(D)}},
\end{aligned}
$$

where we use that $\eta<-N^2$. Then

$$
\sup_{t\le 0} e^{-\eta t}\left\|\int_0^t e^{A(t-s)}P_N\bar{u}^\nu_{tt}(s)\,ds\right\|_{L^2(D)}\le 3\|P_N\bar{u}^\nu_t\|_{C^-_{\eta,L^2(D)}}. \tag{38}
$$

For the higher modes of $\bar{u}^\nu_t$, similarly we have

$$
\begin{aligned}
& \sup_{t\le 0} e^{-\eta t}\left\|\int_{-\infty}^t e^{A(t-s)}(I-P_N)\bar{u}^\nu_{tt}(s)\,ds\right\|_{L^2(D)} \\
\le \;& 3\sup_{t\le 0} e^{-\eta t}\|(I-P_N)\bar{u}^\nu_t(t)\|_{L^2(D)} \\
\le \;& 3\|(I-P_N)\bar{u}^\nu_t\|_{C^-_{\eta,L^2(D)}}.
\end{aligned} \tag{39}
$$

Since $\bar{U}^\nu_t=(\bar{u}^\nu_t,\bar{v}^\nu_t)\in C^-_{\eta,E}$, and by the same discussion as that for Theorem 10, we have

$$
|\hat{u}^\nu|_{C^-_{\eta,L^2(D)}}\to 0\,,\quad \text{as}\quad \nu\to 0\,. \tag{40}
$$

This completes the proof. □

Remark 2. As stationary solutions lie on a random invariant manifold, the above approximation for a random invariant manifold implies that the distribution of stationary solutions to both SWE (1) and SHE (8) coincide with each other. This coincidence was also shown by Cerrai and Freidlin [7] under certain conditions.

A Stationary solutions of linear SWEs and estimates

We give some estimates on the stationary solution$(z^{*\nu}, z_t^{*\nu})$ to the linear SWE (6) and the stationary solution z^* to linear hear equation (7).

The following theorem is classical [7].

Theorem 12. *The stationary solution $z^{*\nu}$ is Gaussian with normal distribution $\mathcal{N}\left(0, \frac{1}{2}A^{-1}Q\right)$ in $L^2(D)$, which is also the distribution of z^* in the space $L^2(D)$.*

We consider $z^{*\nu}$ and z^* on the canonical probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P})$ and the Wiener shift $\{\theta_t\}_{t\in\mathbb{R}}$. Then as stationary solutions to stochastic equations, we write $z^{*\nu}(t) = z^{*\nu}(t,\omega) = z^{*\nu}(\theta_t\omega)$ and $z^*(t) = z^*(t,\omega) = z^*(\theta_t\omega)$.

Theorem 13. *The processes $z^{*\nu}(t,\omega)$ and $z^*(t,\omega)$ satisfy*

$$\lim_{t\to\pm\infty}\frac{1}{t}\|z^{*\nu}(t,\omega)\|_{L^2(D)} = \lim_{t\to\pm\infty}\frac{1}{t}\|z^*(t,\omega)\|_{L^2(D)} = 0$$

for almost all $\omega\in\Omega$.

Proof. The proof is the same as that for scalar systems [12]. ☐

We need an estimate on $\nu z_t^{*\nu}$. Since $\nu^2\mathbb{E}\|z_t^{*\nu}(t)\|^2 = \nu\operatorname{Tr} Q \to 0$, as $\nu\to 0$, then for almost all $\omega\in\Omega$

$$\nu z_t^{*\nu}(t,\omega)\to 0 \quad\text{as}\quad \nu\to 0\,. \tag{41}$$

B Rohlin's classification

Definition 14. *Two random variables X and Y defined on a same probability space $(\Omega,\mathcal{F},\mathbb{P})$ are called equivalent if and only if there is a measurable preserving map $\psi:\Omega\to\Omega$ such that $X(\psi\omega) = Y(\omega)$ for almost all $\omega\in\Omega$.*

If X is equivalent to Y, then X has same distribution as that of Y. Rohlin's result on the classification of the homomorphisms of a Lebesgue space [23] gives an inverse result.

Recall that a homomorphism ψ from probability space $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$ to probability space $(\Omega_2,\mathcal{F}_2,\mathbb{P}_2)$ is a measurable mapping such that $\psi\mathbb{P}_1 = \mathbb{P}_2$. If ψ is measurably invertible, then ψ is an isomorphism. A probability space $(\Omega,\mathcal{F},\mathbb{P})$ is a Lebesgue space [2, Appendix A] if this probability space

is isomorphic to a probability space which is the disjoint union of an at most countable (possibly empty) set $\{\omega_1, \omega_2, \ldots\}$ of points each of positive measure and the space $([0, s), \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the σ-algebra of Lebesgue measurable subsets of the interval $[0, s)$ and λ is the Lebesgue measure. Here $s = 1 - \sum p_n$ where p_n is the measure of the point ω_n. For a measure space, the signature is the mass of its non-atomic part plus the non-increasing sequence of the weights of its atoms.

Rohlin's classification theorem on the homomorphisms of Lebesgue space states the following [23].

Theorem 15. *A homomorphisms of Lebesgue space is determined by the signature of the quotient measure space and the signatures of the condition measure spaces associated with the homomorphism.*

Then we have the following corollary.

Corollary 16. *Random variables X and Y are equivalent if and only if for almost all values taken by these variables, the condition measure spaces are isomorphic; that is, they have the same signature.*

One special case for the above condition measure spaces having the same signature is when almost all conditional measures are purely non-atomic.

The canonical probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ is a Lebesgue space [2]. Now we consider $\eta^{*\nu}(\omega)$ and $\eta^*(\omega)$ which have the same distribution on the same probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. Moreover, the distribution is Gaussian, so almost all conditional measures are purely non-atomic. Then by the above corollary, there is a measure preserving mapping $\psi^\nu : \Omega_0 \to \Omega_0$ such that

$$\eta^*(\psi^\nu \omega) = \eta^{*\nu}(\omega). \tag{42}$$

Acknowledgements This research was supported by the NSFC grant No. 10901083, the ZiJin Foundation of Nanjing University of Science and Technology and by the Australian Research Council grants DP0774311 and DP0988738. The first author was also supported by the CSC to work at the University of Adelaide.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, Boston, 1975.

[2] L. Arnold, *Random Dynamical Systems*, Springer–Verlag, New York/ Berlin, 1998.

[3] V. Barbu, G. Da Prato, The stochastic nonlinear damped wave equation, *Appl. Math. Optim.* **46(2–3)** (2002), 125–141.

[4] D. Blömker & W. Wang, Qualitative properties of local random invariant manifolds for SPDEs with quadratic nonlinearity, *J. Dynam. Diff. Equa.* **22(4)** (2010), 677–695.

[5] P. Brune & B. Schmalfuß, Inertial manifolds for stochastic PDE with dynamical boundary conditions, *Comm. Pure Appl. Anal.* **10(3)** (2011), 831–846.

[6] T. Caraballo, J. Duan, K. Lu & B. Schmalfuß, Invariant manifolds for random and stochastic partial differential equations, *Adv. Nonlinear Stud.* **10(1)** (2010), 23–52.

[7] S. Cerrai & M. Freidlin, On the Smoluchowski–Kramers approximation for a system with an infinite number of degrees of freedom, *Prob. Th. and Relat. Fields* **135** (2006), 363–394.

[8] S. Cerrai & M. Freidlin, Smoluchowski–Kramers approximation for a general class of SPDE's, *J. Evol. Equa.* **6** (2006), 657–689.

[9] S. Chow & K. Lu, Invariant manifolds for flows in Banach spaces, *J. Diff. Equa.* **74** (1988), 285–317.

[10] G. Da Prato, D. Gatarek & J. Zabczyk, Invariant measures for semilinear stochastic equations, *Stoch. Anal. Appl.* **10(4)** (1992), 387–408.

[11] G. Da Prato & J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, 1996.

[12] J. Duan, K. Lu & B. Schmalfuß, Invariant manifolds for stochastic partial differential equations, *Ann. of Prob.* **31** (2003), 2109–2135.

[13] J. Duan, K. Lu & B. Schmalfuß, Smooth stable and unstable manifolds for stochastic evolutionary equations, *J. Dynam. Differential Equations* **16** (2004), 949–972.

[14] O. van Gaans & S. Verduyn Lunel, Long term behavior of dichotomous stochastic differential equations in Hilbert spaces, *Commun. Contemp. Math.* **6(3)** (2004), 349–376.

[15] M. Hairer, Exponential mixing properties of stochastic PDEs through asymptotic coupling, *Probab. Theory Relat. Fields* **124(3)** (2002), 345–380.

[16] Z. Liu, Stochastic inertial manifolds for damped wave equations, *Stoch. Dyn.* **10(2)** (2010), 211–230.

[17] K. Lu & Schmalfuß, Invariant manifolds for stochastic wave equations, *J. Diff. Equa.* bf 236(2) (2007), 460–492.

[18] Y. Lv & W. Wang, Limit dynamics for stochastic wave equations, *J. Diff. Equa.* **244** (2008),1–23.

[19] X. Mora, Finite-dimensional attracting invariant manifolds for damped semilinear wave equations, *Contributions to Nonlinear Partial Differential Equations* **II** (Paris,1985), 172–183. *Pitman Res. Notes Math. Ser.,* **155**, Longman Sci. Tech., Harlow, 1987.

[20] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, *Applied Mathematical Sciences*, Vol. 44, Springer–Verlag, New York, 1983.

[21] A. J. Roberts. Resolving the multitude of microscale interactions accurately models stochastic partial differential equations. *LMS J. Computation and Maths*, 9:193–221, 2006. `http://www.lms.ac.uk/jcm/9/lms2005-032`.

[22] A. J. Roberts. Normal form transforms separate slow and fast modes in stochastic dynamical systems. *Physica A*, 387:12–38, 2008. `http://dx.doi.org/10.1016/j.physa.2007.08.023`.

[23] V. A. Rokhlin, (1962), On the fundamental ideas of measure theory, *Tran. Amer. Math. Soc.* **1(10)** (1962), 1–54.

[24] W. Wang & Y. Lv, Limit behavior of nonlinear stochastic wave equations with singular perturbation, *Disc. Cont. Dyna. Syst. B* **13(1)** 2010, 175–193.

[25] W. Wang & J. Duan, A dynamical approximation for stochastic partial differential equations, *J. Math. Phys.* **48(10)**(2007), 102701.

[26] W. Wang and A. J. Roberts. Macroscopic discrete modelling of stochastic reaction-diffusion equations on a periodic domain. *Discrete and Continuous Dynamical System A*, 31(1):253–273, 2011.

[27] W. Wang and A. J. Roberts. Slow manifold and averaging for slow fast stochastic differential system. Technical report, `http://arxiv.org/abs/0903.1375`, 2009.